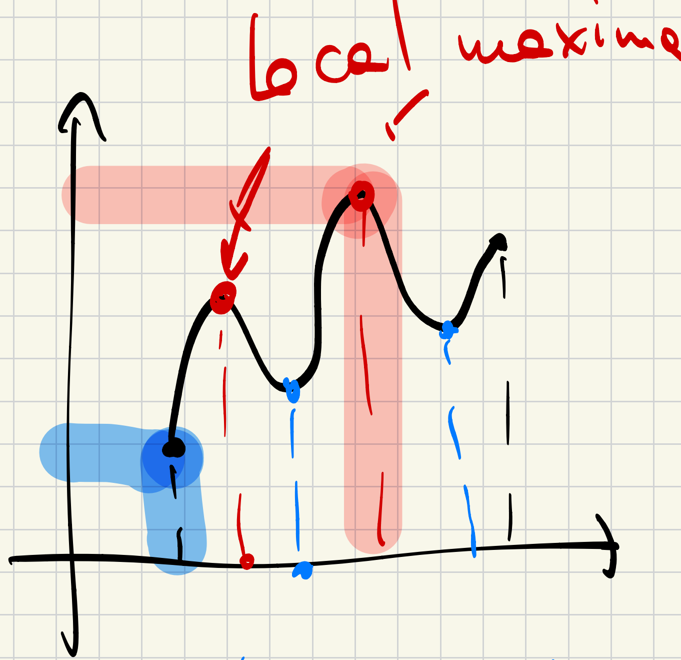
Analysis I Lecture 21

Last time:

- Local global extreme
- Stationary points
- Rolle's theorem
- Mean value theorem




$x_0$ s.t. $f'(x_0)=0$

Thm (MVT) $f : [a,b] \to \mathbb{R}$ cont.

and differentiable on $(a,b)$

then $\exists\, c \in (a,b)$ s.t.

$$\boxed{f'(c) = \frac{f(b) - f(a)}{b-a}}$$

$$\frac{f(b) - f(a)}{b-a} = 0$$

$\Uparrow$

Rolle's partial case when $\underline{\underline{f(a) = f(b)}}$,

## Today:

- Monotonicity and differentiation
- L'Hôpital rule
- Taylor expansion.

# Monotonicity of differentiable functions

**Corollary 7.62** Let $f:[a,b] \to \mathbb{R}$ be continuous and diff on $(a,b)$. Then,

1) $f$ is increasing (decreasing) if and only if $f'(x) \geq 0$ $(f'(x) \leq 0)$ $\forall x \in (a,b)$

2) if $f'(x) > 0$ $(f'(x) < 0)$ $\forall x \in (a,b)$ then $f$ is strictly increasing (decreasing).

Example ⚠ stricly increasing $\not\Rightarrow$ $f'(x)>0$

Eg $f(x) = x^3$ stricly monotone function

but $f'(x) = 3 \cdot x^2$

$f'(0) = 0$ not stricly positive.

E.g $f(x) = \cos(x)$

$$f'(x) = -\sin(x) < 0 \quad \text{for } x \in (0, \pi)$$

$$> 0 \quad \text{for } x \in (\pi, 2\pi)$$

$\Rightarrow$ $\cos(x)$ is stricly decreasing on $(0, \pi)$.

and is stricly increasing on $(\pi, 2\pi)$.

E.g. $f(x) = 2x^3 - 3x^2 + 6x + 1$

When $f(x)$ is increasing and decreasing?

$$f'(x) = 6x^2 - 6x + 6 = 6(x^2 - x + 1)$$

$x^2 - x + 1 > 0$ for all $x \in \mathbb{R}$

(since discriminant $< 0$ and quadratic coeff. is positive)

$\Rightarrow f$ is stricly increasing on $\mathbb{R}$.

# L'Hôpital rule

How to compute $\lim\limits_{x \to \infty} \frac{e^x}{x}$ ?

$e^x \to +\infty$

$x \to +\infty$

**Idea:** Instead we can compute

$$\lim_{x \to \infty} \frac{(e^x)'}{x'} = \lim_{x \to \infty} \frac{e^x}{1} = +\infty$$

**Theorem 7.67** $f, g : I \to \mathbb{R}$ - differentiable

and let $x_0 \in I$ s.t.

- $g(x) \neq 0$, $g'(x) \neq 0$ (in a punctured neighborhood of $x_0$)

- $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0$ or $\pm\infty$

Then $\boxed{\text{if } \lim_{x \to x_0} \frac{f'(x)}{g'(x)} \text{ exists we have: } \lim_{x \to x_0} \frac{f}{g} = \lim_{x \to x_0} \frac{f'}{g'}.}$

# Remark

1. Can have $x_0 = \pm\infty$

2. Also works for left, right limits:

If $\lim_{x \to x_0^+} \frac{f'(x)}{g'(x)}$ exists then it is equal to $\lim_{x \to x_0^+} \frac{f(x)}{g(x)}$.

Example $\lim\limits_{x\to+\infty} \frac{e^x}{x}$

$$\lim_{x\to\infty} e^x = \lim_{x\to\infty} x = +\infty$$

$$\lim_{x\to\infty} \frac{(e^x)'}{x'} = \lim_{x\to\infty} \frac{e^x}{1} = +\infty$$

$$\Rightarrow \lim_{x\to+\infty} \frac{e^x}{x} = +\infty .$$

Remark we can Apply L'Hôpital several times:

$$\lim_{x\to\infty} \frac{e^x}{x^n} = \underbrace{\lim_{x\to\infty} \frac{(e^x)'}{(x^n)'}}_{\text{If exists}} = \lim_{x\to\infty} \frac{e^x}{n \cdot x^{n-1}}$$

$\nearrow \infty$

$\searrow \infty$

$$= \underbrace{\lim_{x\to\infty} \frac{(e^x)'}{(n \cdot x^{n-1})'}}_{\text{if Exists}} = \lim_{x\to\infty} \frac{e^x}{n(n-1) \cdot x^{n-2}}$$

After n-th application of L'Hôpital we get:

$$\lim_{x\to\infty}\frac{e^x}{x^n} = \lim_{x\to\infty}\frac{(e^x)^{(n)}}{(x^n)^{(n)}}$$ if the right limit exists.

$$\lim_{x\to\infty}\frac{e^x}{n!} = +\infty .$$

Important to check that

$$\lim_{x \to x_0} \frac{f'}{g'} \quad \text{exists:}$$

E.g. 1) $f = x + \sin(x)$ $\qquad g(x) = x$

$$\lim_{x \to \infty} \frac{x + \sin(x)}{x} = \lim_{x \to \infty} \left(1 + \frac{\sin x}{x}\right) = 1$$

$$\lim_{x \to \infty} \frac{f'}{g'} = \lim_{x \to \infty} \frac{1 + \cos(x)}{1} = \lim_{x \to \infty} \cos(x) \text{ doesn't exist}$$

2) $f(x) = \sqrt{x} + \sin x \qquad g(x) = x$

$$\lim_{x \to \infty} \frac{\sqrt{x} + \sin x}{x} = \lim_{x \to \infty} \left( \frac{1}{\sqrt{x}} + \frac{\sin(x)}{x} \right) = 0$$

$$\lim_{x \to \infty} \frac{f'}{g'} = \lim_{x \to \infty} \frac{\frac{1}{2\sqrt{x}} + \cos(x)}{1}$$ doesn't exists.

# Taylor expansion

**Idea:** Approximate functions by polynomials.

Example : Linear approximation

$$f(x) = \underbrace{\overset{f(x_0)}{\cancel{a_0}} + a_1(x - x_0)}_{\text{linear function}} + \underbrace{r(x)}_{\text{remainder}}$$

$a_0 \, , \, a_1 \in \mathbb{R}$

$$f(x_0) = a_0 + 0 + \underset{0}{\underset{\shortparallel}{r(0)}} \Rightarrow a_0 = f(x_0)$$

want reminder to be small

w.r.t. linear function approximating

function $f$:

- $r(x_0) = 0 \Rightarrow a_0 = f(x_0)$

- $\boxed{\lim_{x \to x_0} \frac{r(x)}{x - x_0} = 0}$ "$r(x)$ does not change linear part of $f$"

How to find $a_1$?

$$f(x) = f(x_0) + a_1(x - x_0) + r(x)$$

$$\Rightarrow \quad a_1 = \lim_{x \to x_0}\left(\frac{f(x) - f(x_0)}{x - x_0} - \frac{r(x)}{x - x_0}\right)$$

$$= \underbrace{\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}}_{f'(x_0)} - \underbrace{\cancel{\lim_{x \to x_0} \frac{r(x)}{x - x_0}}}_{0} = f'(x_0)$$

In total if we want to get linear approximation (Also called linear expension) for $f(x)$:

$$f(x) = \underline{\underline{f(x_0)}} + \underline{\underline{f'(x_0)}}(x - x_0) + r(x)$$

<u>Definition</u> 7.72 $f: E \to \mathbb{R}$ admits an expansion of order $n$ at $x_0$ if there is an equality of the form:

$$f(x) = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \ldots + a_n (x - x_0)^n + \underbrace{(x - x_0)^n \varepsilon_n (x)}_{r_n(x)}$$

Where $\varepsilon_n : E \to \mathbb{R}$ is a function with

$$\lim_{x \to x_0} \varepsilon_n = 0.$$

Here $(x-x_0)^n \cdot \varepsilon_n(x) = r_n(x)$

so we get $\boxed{\dfrac{r_n(x)}{(x-x_0)^n} = \varepsilon_n(x) \to 0}$

**Proposition 7.73** If $n$th order approximation of $f$ exists it is unique.

That is if

$$f(x) = a_0 + a_1(x-x_0) + \dots + a_n(x-x_0)^n + (x-x_0)^n \cdot \varepsilon_n(x)$$

$$= b_0 + b_1(x-x_0) + \dots + b_n(x-x_0)^n + (x-x_0)^n \cdot \varepsilon_n'(x)$$

Then $a_i = b_i$ and $\varepsilon_n(x) = \varepsilon_n'(x)$

## When and How can we find expansions?

**Theorem 7.74.** Let $f: I \to \mathbb{R}$ be $C^{k+1}$ and let $x_0 \in I$. Then for each $x \in I$ there exists $x' \in (x, x_0)$ if $x < x_0$ ($x' \in (x_0, x)$ if $x > x_0$)

s.t.

$$\boxed{f(x) = \sum_{i=0}^{n} \underbrace{\frac{f^{(i)}(x_0)}{i!}}_{= a_i} (x - x_0)^i + f^{(n+1)}(x') \frac{(x - x_0)^{n+1}}{(n+1)!}}$$

In particular we get:

$$\boxed{f(x) = \sum_{j=0}^{n} \underbrace{\frac{f^{(j)}(x_0)}{j!}}_{a_j} (x-x_0)^j + \underbrace{(x-x_0)^n \cdot \varepsilon_n}_{\text{with } \lim_{x \to x_0} \varepsilon_n = 0}}$$

Order $n$- expansion of $f$

__Example__ $e^x$ Expansion at $x_0=0$

Since $e^x \in C^\infty(\mathbb{R})$ we have

Expansion of any order:

$$e^x = \sum_{i=0}^{n} \underbrace{\frac{(e^x)^{(i)}(0)}{i!}} \cdot (x-0)^i$$

$$+ (x-0)^n \cdot \varepsilon_n$$

$$(e^x)^{(i)}(0) = e^x(0) = e^0 = 1$$

$$\Rightarrow \boxed{e^x = \sum_{i=0}^{n} \frac{1}{i!} \cdot x^i + x^n \cdot \varepsilon_n(x)}$$

order n- expansion of $e^x$.

Example $\quad \frac{1}{1-x} = f(x) \quad$ at $\quad x_0 = 0$

$$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}} \Rightarrow f^{(n)}(0) = n!$$

$$f'(x) = -\frac{1}{(1-x)^2} \cdot (1-x)' = \frac{1}{(1-x)^2}$$

$$f''(x) = \frac{-2}{(1-x)^3} \cdot (1-x)' = \frac{2}{(1-x)^{3}} \quad 3 = 2+1$$

$\vdots$

So we get:

$f^{(i)}(0)$

$$\frac{1}{1-x} = \sum_{i=0}^{n} \frac{i!}{i!} \cdot x^i + x^n \cdot \varepsilon_n$$

$$= \boxed{\sum_{i=0}^{n} x^i + x^n \cdot \varepsilon_n}$$

Order n- expansion

Example:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + x^n \varepsilon_n$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + x^n \cdot \varepsilon_n$$

Expansions of products sums and compositions:

If $$f(x) = \sum_{i=0}^{n} a_i (x-x_0)^i + (x-x_0)^n \cdot \varepsilon_n$$

$$g(x) = \sum_{i=0}^{n} b_i (x-x_0)^i + (x-x_0)^n \cdot \varepsilon_n'$$

$$\Rightarrow (f+g)(x) = \sum_{i=0}^{n} (a_i+b_i)(x-x_0)^i + (x-x_0)^n \varepsilon_n''$$

$$f \cdot g(x) = \overbrace{\left( \sum_{i=0}^{n} a_i (x - x_0)^i + \varepsilon'_n \cdot (x - x_0)^n \right)}^{f} \times$$

$$\times \underbrace{\left( \sum_{i=0}^{n} b_i (x - x_0)^i + \varepsilon''_n (x - x_0)^n \right)}_{g}$$

$$=$$

$$= \left( \sum a_i (x-x_0)^i \right) \left( \sum b_i (x-x_0)^i \right)$$

$$+ (x - x_0)^n \underbrace{\left( \varepsilon' \cdot \sum b_i (x-x_0)^i + \left( \sum a_i (x-x_0)^i \right) \cdot \varepsilon'' + (x-x_0)^n \varepsilon' \cdot \varepsilon'' \right)}_{\to 0 \text{ as } x \to x_0}$$

To find order $n$ expansion of $f \cdot g$ we only need to compute terms in

$$\left(\sum_{i=0}^{n} a_i (x-x_0)^i\right) \cdot \left(\sum_{j=0}^{n} b_j (x-x_0)^j\right)$$

of order $\leq n$.

Example: Assume

$$f = 1 + 2x - 3x^2 + \varepsilon'(x) \cdot x^2$$

$$g = 1 + x^2 + \varepsilon''(x) \cdot x^2$$

What is order 2 expansion of $f \cdot g$?

$$f \cdot g = \underbrace{\left(1 + 2x - 3x^2\right)\left(1 + x^2\right)} + \varepsilon(x) \cdot x^2$$

$$\underbrace{(1+2x-3x^2)(1+x^2)}=$$

$$= 1 + x^2 + 2x + 2x^3$$

$$-3x^2 - 3x^4 =$$

deg > 2

$$= 1 + 2x - 2x^2 + \varepsilon \cdot x^2 \Rightarrow$$

$$f \cdot g = 1 + 2x - x^2 + \varepsilon \cdot x^2.$$